EE 703: Digital Message Transmission (Autumn 2020)
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Midsem Exam: 20 points — Date: October 5, 2020

1. [4 points] Let $a_p(t)$ and $b_p(t)$ be two real passband signals whose corresponding Fourier transforms satisfy

$$\begin{aligned} A_p(f) = 0 \quad & \text{for } f \notin [f_c - W, f_c + W] \cup [-f_c - W, -f_c + W], \\ B_p(f) = 0 \quad & \text{for } f \notin [f_c - W, f_c + W] \cup [-f_c - W, -f_c + W], \end{aligned}$$

for some center frequency $f_c > 0$ and constant $W > 0$ where $f_c > W$.

Let $a(t) = a_c(t) + ja_s(t)$ be the complex baseband representation of $a_p(t)$. Let $b(t) = b_c(t) + jb_s(t)$ be the complex baseband representation of $b_p(t)$.

(a) Prove that the following equations hold.

$$\int_{-\infty}^{\infty} a_c(t) b_s(t) \cos 2\pi f_c t \sin 2\pi f_c t \; dt = 0,$$

$$\int_{-\infty}^{\infty} a_s(t) b_c(t) \cos 2\pi f_c t \sin 2\pi f_c t \; dt = 0.$$

(b) Use the above result to prove that

$$\langle a_p, b_p \rangle = \langle a_c, b_c \rangle + \langle a_s, b_s \rangle .$$

2. [4 points] Let $\phi_1(t), \phi_2(t), \phi_3(t)$ be real unit energy signals which are orthogonal, i.e. $\|\phi_1\|^2 = \|\phi_2\|^2 = \|\phi_3\|^2 = 1$ and $\langle \phi_1, \phi_2 \rangle = \langle \phi_2, \phi_3 \rangle = \langle \phi_3, \phi_1 \rangle = 0$.

Determine an orthonormal basis for the set of signals $s_1(t), s_2(t), s_3(t)$ which are given by the following equations where $j = \sqrt{-1}$.

$$\begin{aligned} s_1(t) &= \phi_1(t) + j\phi_2(t), \\ s_2(t) &= \phi_2(t) + j\left[\phi_3(t) - \phi_1(t)\right] \\ s_3(t) &= \phi_3(t). \end{aligned}$$

3. [4 points] Let $\phi_1(t)$ and $\phi_2(t)$ be real unit energy signals which are orthogonal, i.e. $\|\phi_1\|^2 = \|\phi_2\|^2 = 1$ and $\langle \phi_1, \phi_2 \rangle = 0$. Let $\psi_1(t)$ and $\psi_2(t)$ be given by

$$\begin{aligned} \psi_1(t) &= \frac{1}{\sqrt{2}} \left[\phi_1(t) + \phi_2(t)\right], \\ \psi_2(t) &= \frac{1}{\sqrt{2}} \left[\phi_1(t) - \phi_2(t)\right]. \end{aligned}$$

Consider the following hypothesis testing problem in AWGN where the hypotheses are equally likely, $s_1(t) = 2\psi_1(t) + 4\psi_2(t)$, and $s_2(t) = \psi_1(t) + 3\psi_2(t)$.

$$\begin{aligned} H_1 &: \; y(t) = s_1(t) + n(t) \\ H_2 &: \; y(t) = s_2(t) + n(t) \end{aligned}$$

Let the observed signal be given by $y(t) = \frac{3}{2}\psi_1(t) + \psi_2(t)$**. Find the output of the optimal decision rule.** You have to show the calculations which lead to your answer.

4. [4 points] Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

$$\begin{aligned} H_0 &: Y \sim U[a, b] \\ H_1 &: Y \sim U[c, d] \end{aligned}$$

where U denotes the uniform distribution and a, b, c, d are real numbers satisfying $a < c < d < b$.

(a) Derive the optimal decision rule.

(b) Find the decision error probability of the optimal decision rule.

5. [4 points] Suppose X_1, X_2, X_3 are jointly Gaussian random variables each having mean $\mu > 0$ and variance $\sigma^2 > 0$. We are also given that X_1, X_2, X_3 are independent random variables.

Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

$$H_1 : \mathbf{Y} = \begin{bmatrix} X_1 + X_2 \\ X_2 + X_3 \end{bmatrix}$$

$$H_2 : \mathbf{Y} = \begin{bmatrix} X_1 - X_2 \\ X_2 - X_3 \end{bmatrix}$$

(a) [3 points] **Derive the optimal decision rule**. Show your steps and simplify the rule as much as possible. The answers to the next three parts will not be considered if the answer to this part is incorrect.

(b) [$\frac{1}{3}$ points] What is the decision of the optimal decision rule if $\mathbf{y} = \begin{bmatrix} \mu & \mu \end{bmatrix}^T$? Explain your answer.

(c) [$\frac{1}{3}$ points] What is the decision of the optimal decision rule if $\mathbf{y} = \begin{bmatrix} \mu & -\mu \end{bmatrix}^T$? Explain your answer.

(d) [$\frac{1}{3}$ points] What is the decision of the optimal decision rule if $\mathbf{y} = \begin{bmatrix} 0 & 2\mu \end{bmatrix}^T$? Explain your answer.