Complex Baseband Representation

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Complex Baseband Representation

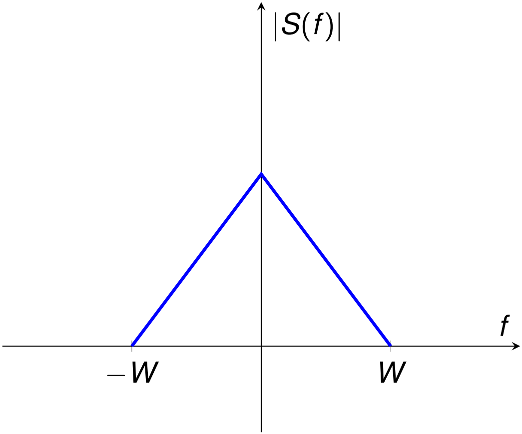
- Contains all the information of a real-valued passband signal
- Requires a smaller sampling rate for discrete-time representation
- Enables modular transceiver design
  - Signal processing algorithms are implemented in the baseband
  - Carrier frequency can be chosen independently

Baseband Signals

A signal $s(t)$ is said to be *baseband* if

$$S(f) \approx 0, \quad |f| > W$$

for some $W > 0$

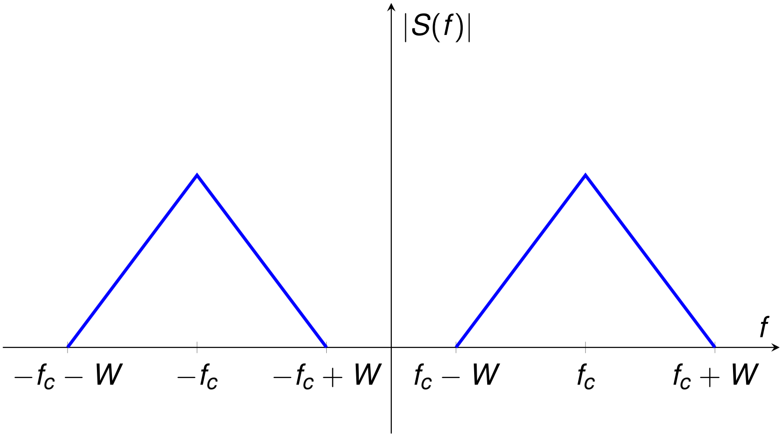


Passband Signals

A signal $s(t)$ is said to be *passband* if

$$S(f) \approx 0, \quad |f \pm f_c| > W,$$

where $f_c > W > 0$



Sampling Theorem

Theorem

If a signal $s(t)$ is bandlimited to B,

$$S(f) = 0, \quad |f| > B$$

then a sufficient condition for exact reconstructability is a uniform sampling rate f_s where

$$f_s > 2B.$$

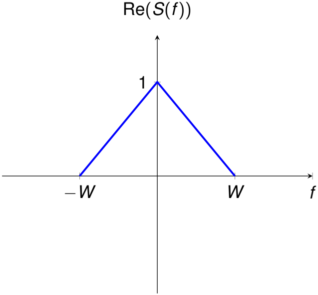
Baseband Signals $B = W$

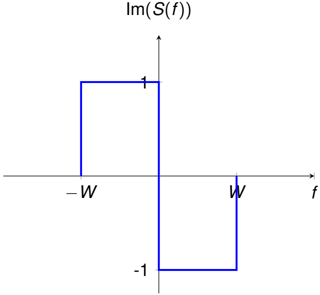
Passband Signals $B = f_c + W$

Can we reduce the sampling rate for passband signals?
Yes. By using the complex baseband representation.

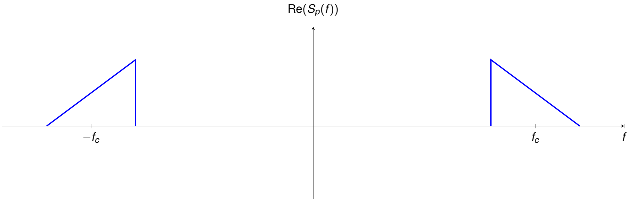
Fourier Transform for Real Signals

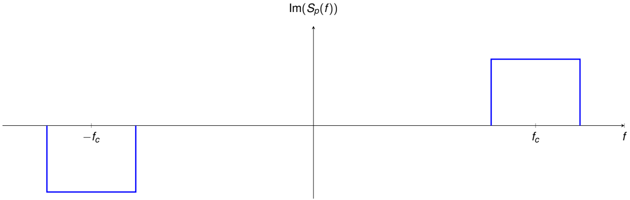
$$\mathrm{Im}[s(t)] = 0 \quad \Rightarrow \quad S(f) = S^*(-f)$$
$$\Rightarrow \quad \mathrm{Re}\,(S(f)) = \mathrm{Re}\,(S(-f)),$$
$$\mathrm{Im}\,(S(f)) = -\,\mathrm{Im}\,(S(-f))$$



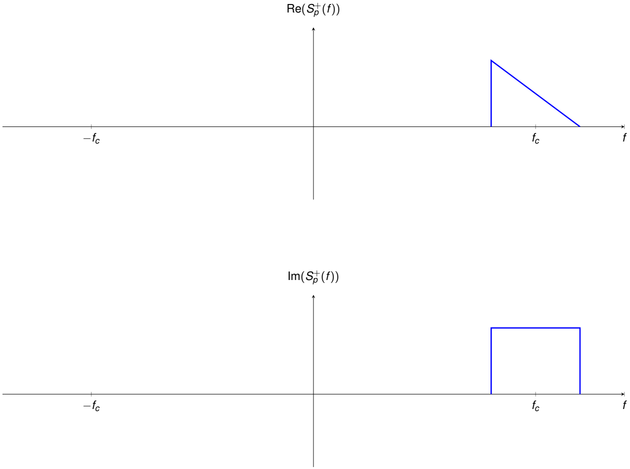


Fourier Transform of a Real Passband Signal



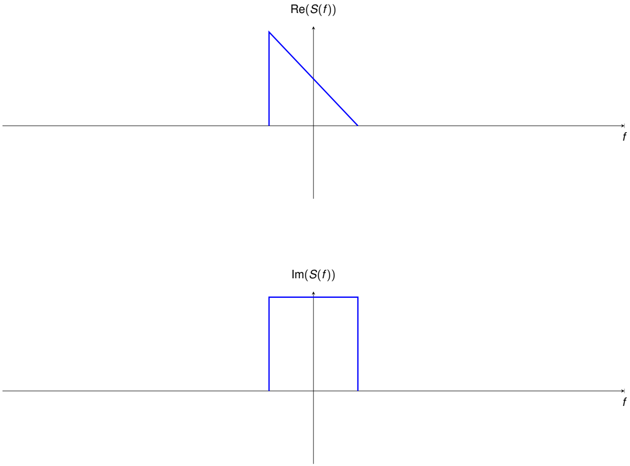


Positive Spectrum of a Real Passband Signal



$$S_p^+(f) = S_p(f)u(f)$$

Complex Envelope of a Real Passband Signal



$$S(f) = \sqrt{2} S_p^+(f + f_c) = \sqrt{2} S_p(f + f_c) u(f + f_c)$$

Complex Envelope in Time Domain

Frequency Domain Representation

$$S(f) = \sqrt{2}S_p^+(f+f_c) = \sqrt{2}S_p(f+f_c)u(f+f_c)$$

Time Domain Representation of Positive Spectrum

$$\begin{aligned} S_p^+(f) &= S_p(f)u(f) \\ s_p^+(t) &= s_p(t) \star \mathcal{F}^{-1}\left[u(f)\right] \end{aligned}$$

Time Domain Representation of Frequency Domain Unit Step

$$\begin{aligned} u(t) &\leftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \\ \frac{1}{j2\pi t} + \frac{1}{2}\delta(t) &\leftrightarrow u(-f) \\ \frac{j}{2\pi t} + \frac{1}{2}\delta(t) &\leftrightarrow u(f) \end{aligned}$$

Complex Envelope in Time Domain

Time Domain Representation of Positive Spectrum

$$
\begin{aligned}
s_p^+(t) &= s_p(t) \star \left[\frac{1}{2}\delta(t) + \frac{j}{2\pi t}\right] \\
&= \frac{1}{2}\left[s_p(t) + j\hat{s}_p(t)\right]
\end{aligned}
$$

Time Domain Representation of Complex Envelope

$$
\begin{aligned}
\sqrt{2}S_p(f)u(f) &\leftrightarrow \frac{1}{\sqrt{2}}\left[s_p(t) + j\hat{s}_p(t)\right] \\
\sqrt{2}S_p(f+f_c)u(f+f_c) &\leftrightarrow \frac{1}{\sqrt{2}}\left[s_p(t) + j\hat{s}_p(t)\right]e^{-j2\pi f_c t} \\
S(f) &\leftrightarrow \frac{1}{\sqrt{2}}\left[s_p(t) + j\hat{s}_p(t)\right]e^{-j2\pi f_c t} \\
s(t) &= \frac{1}{\sqrt{2}}\left[s_p(t) + j\hat{s}_p(t)\right]e^{-j2\pi f_c t}
\end{aligned}
$$

Passband Signal in terms of Complex Envelope

Complex Envelope

$$s(t) = s_c(t) + js_s(t)$$

$s_c(t)$ In-phase component

$s_s(t)$ Quadrature component

Time Domain Relationship

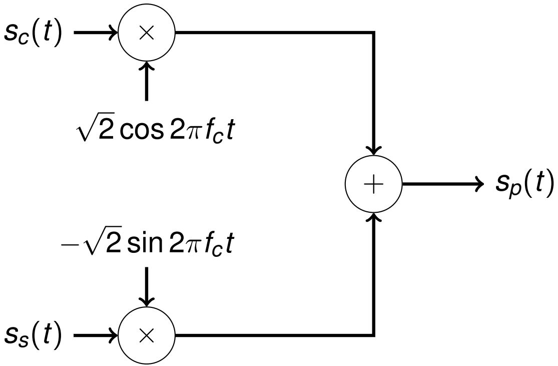
$$\begin{aligned} s_p(t) &= \text{Re}\left[\sqrt{2}s(t)e^{j2\pi f_c t}\right] \\ &= \text{Re}\left[\sqrt{2}\{s_c(t) + js_s(t)\}e^{j2\pi f_c t}\right] \\ &= \sqrt{2}s_c(t)\cos 2\pi f_c t - \sqrt{2}s_s(t)\sin 2\pi f_c t \end{aligned}$$

Frequency Domain Relationship

$$S_p(f) = \frac{S(f - f_c) + S^*(-f - f_c)}{\sqrt{2}}$$

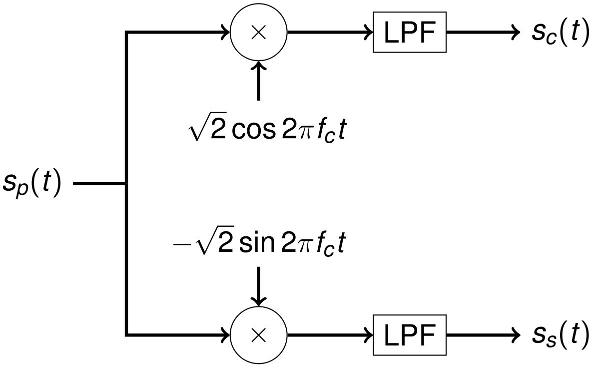
Upconversion

$$s_p(t) = \sqrt{2} s_c(t) \cos 2\pi f_c t - \sqrt{2} s_s(t) \sin 2\pi f_c t$$



Downconversion

$$
\begin{aligned}
\sqrt{2} s_p(t) \cos 2\pi f_c t &= 2 s_c(t) \cos^2 2\pi f_c t - 2 s_s(t) \sin 2\pi f_c t \cos 2\pi f_c t \\
&= s_c(t) + s_c(t) \cos 4\pi f_c t - s_s(t) \sin 4\pi f_c t
\end{aligned}
$$



Inner Product and Energy

Let $s(t)$ and $r(t)$ be signals. Recall the following definitions

- Inner Product

$$\langle s, r \rangle = \int_{-\infty}^{\infty} s(t) r^*(t) \, dt$$

- Parseval's identity

$$\int_{-\infty}^{\infty} s(t) r^*(t) \, dt = \int_{-\infty}^{\infty} S(f) R^*(f) \, df$$

- Energy

$$E_s = \|s\|^2 = \langle s, s \rangle = \int_{-\infty}^{\infty} |s(t)|^2 \, dt$$

I and Q Components of a Passband Signal

$$s_p(t) = \underbrace{\sqrt{2} s_c(t) \cos 2\pi f_c t}_{I \text{ Component}} - \underbrace{\sqrt{2} s_s(t) \sin 2\pi f_c t}_{Q \text{ Component}}$$

$$\begin{aligned} x_i(t) &= \sqrt{2} s_c(t) \cos 2\pi f_c t \\ x_q(t) &= \sqrt{2} s_s(t) \sin 2\pi f_c t \end{aligned}$$

I and Q Components of a Passband Signal are Orthogonal

$$\langle x_i, x_q \rangle = 0$$

Passband and Baseband Inner Products

$$\langle u_p, v_p \rangle = \langle u_c, v_c \rangle + \langle u_s, v_s \rangle = \text{Re}\left(\langle u, v \rangle\right)$$

Energy of Complex Envelope = Energy of Passband Signal

$$\|s\|^2 = \|s_p\|^2$$

Complex Baseband Equivalent of Passband Filtering

$s_p(t)$ Passband signal
$h_p(t)$ Impulse response of passband filter
$y_p(t)$ Filter output

$$\begin{aligned} y_p(t) &= s_p(t) \star h_p(t) \\ Y_p(f) &= S_p(f) H_p(f) \end{aligned}$$

$$\begin{aligned} S_+(f) &= S_p(f) u(f) \\ H_+(f) &= H_p(f) u(f) \\ Y_+(f) &= Y_p(f) u(f) \\ Y_+(f) &= S_+(f) H_+(f) \end{aligned}$$

$$Y(f) = \sqrt{2} Y_+(f + f_c) = \sqrt{2} S_+(f + f_c) H_+(f + f_c) = \frac{1}{\sqrt{2}} S(f) H(f)$$

Complex Baseband Equivalent of Passband Filtering

$$
\begin{aligned}
y(t) &= \frac{1}{\sqrt{2}} s(t) \star h(t) \\
y_c &= \frac{1}{\sqrt{2}} (s_c \star h_c - s_s \star h_s) \\
y_s &= \frac{1}{\sqrt{2}} (s_s \star h_c + s_c \star h_s)
\end{aligned}
$$

References

- Section 2.2, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008