PSD of Digitally Modulated Signals

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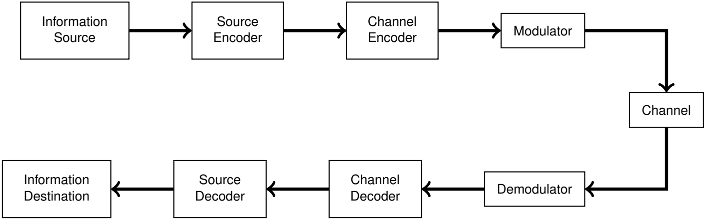
Digital Modulation

Digital Modulation

Definition

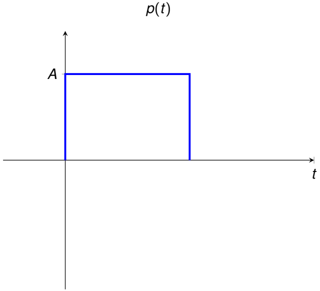
The process of mapping a bit sequence to signals for transmission over a channel.

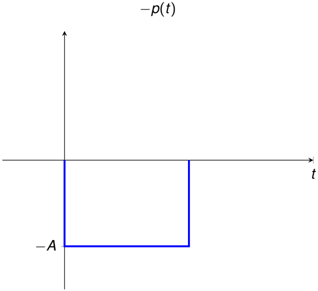


Digital Modulation

Example (Binary Baseband PAM)

$1 \rightarrow p(t)$ and $0 \rightarrow -p(t)$





Classification of Modulation Schemes

- Memoryless
  - Divide bit sequence into k-bit blocks
  - Map each block to a signal $s_m(t), \quad 1 \leq m \leq 2^k$
  - Mapping depends only on current k-bit block
- Having Memory
  - Mapping depends on current k-bit block and $L - 1$ previous blocks
  - L is called the constraint length
- Linear
  - Complex baseband representation of transmitted signal has the form
    $$u(t) = \sum_n b_n g(t - nT)$$
    where b_n's are the transmitted symbols and g is a fixed baseband waveform
- Nonlinear

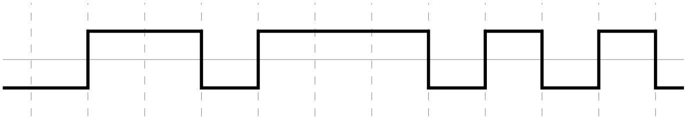
PSD Definition for Linearly Modulated Signals

PSD Definition for Linearly Modulated Signals

- Consider a real binary PAM signal

$$u(t) = \sum_{n=-\infty}^{\infty} b_n g(t - nT)$$

where $b_n = \pm 1$ with equal probability and $g(t)$ is a baseband pulse of duration T



- ~~PSD $= \mathcal{F}[R_u(\tau)]$~~ Neither SSS nor WSS

Cyclostationary Random Process

Definition (Cyclostationary RP)

A random process $X(t)$ is cyclostationary with respect to time interval T if it is statistically indistinguishable from $X(t - kT)$ for any integer k.

Definition (Wide Sense Cyclostationary RP)

A random process $X(t)$ is wide sense cyclostationary with respect to time interval T if the mean and autocorrelation functions satisfy

$$\begin{aligned} m_X(t) &= m_X(t - T) \quad \text{for all } t, \\ R_X(t_1, t_2) &= R_X(t_1 - T, t_2 - T) \quad \text{for all } t_1, t_2. \end{aligned}$$

Power Spectral Density of a Cyclostationary Process

To obtain the PSD of a cyclostationary process with period T

- Calculate autocorrelation of cyclostationary process $R_X(t, t-\tau)$
- Average autocorrelation between 0 and T, $R_X(\tau) = \frac{1}{T}\int_0^T R_X(t, t-\tau)\, dt$
- Calculate Fourier transform of averaged autocorrelation $R_X(\tau)$

Power Spectral Density of a Realization

Time windowed realizations have finite energy

$$
\begin{aligned}
x_{T_o}(t) &= x(t) I_{[-\frac{T_o}{2}, \frac{T_o}{2}]}(t) \\
S_{T_o}(f) &= \mathcal{F}(x_{T_o}(t)) \\
\hat{S}_x(f) &= \frac{|S_{T_o}(f)|^2}{T_o} \quad \text{(PSD Estimate)}
\end{aligned}
$$

PSD of a realization

$$
\bar{S}_x(f) = \lim_{T_o \to \infty} \frac{|S_{T_o}(f)|^2}{T_o}
$$

$$
\frac{|S_{T_o}(f)|^2}{T_o} \leftrightarrow \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u) x_{T_o}^*(u - \tau)\, du = \hat{R}_x(\tau)
$$

Power Spectral Density of a Cyclostationary Process

$X(t)X^*(t-\tau) \sim X(t+T)X^*(t+T-\tau)$ for cyclostationary $X(t)$

$$
\begin{aligned}
\hat{R}_x(\tau) &= \frac{1}{T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x(t)x^*(t-\tau)\,dt \\
&= \frac{1}{KT}\int_{-\frac{KT}{2}}^{\frac{KT}{2}} x(t)x^*(t-\tau)\,dt \qquad (\text{for } T_o = KT) \\
&= \frac{1}{T}\int_0^T \frac{1}{K}\sum_{k=-\frac{K}{2}}^{\frac{K}{2}-1} x(t+kT)x^*(t+kT-\tau)\,dt \qquad (\text{for even } K) \\
&\underset{K\to\infty}{\longrightarrow} \frac{1}{T}\int_0^T E\left[X(t)X^*(t-\tau)\right]\,dt \\
&= \frac{1}{T}\int_0^T R_X(t,t-\tau)\,dt = R_X(\tau)
\end{aligned}
$$

PSD of a cyclostationary process = $\mathcal{F}[R_X(\tau)]$

PSD of a Linearly Modulated Signal

- Consider

$$u(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

- $u(t)$ is cyclostationary wrt to T if $\{b_n\}$ is stationary
- $u(t)$ is wide sense cyclostationary wrt to T if $\{b_n\}$ is WSS
- Suppose $R_b[k] = E[b_n b_{n-k}^*]$
- Let $S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k] z^{-k}$
- The PSD of $u(t)$ is given by

$$S_u(f) = S_b\left(e^{j2\pi fT}\right) \frac{|P(f)|^2}{T}$$

PSD of a Linearly Modulated Signal

$$
\begin{aligned}
R_u(\tau) & \\
&= \frac{1}{T}\int_0^T R_u(t+\tau, t)\, dt \\
&= \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left[b_n b_m^* p(t-nT+\tau)p^*(t-mT)\right]\, dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-mT}^{-(m-1)T} E\left[b_{m+k} b_m^* p(\lambda-kT+\tau)p^*(\lambda)\right]\, d\lambda \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[b_{m+k} b_m^* p(\lambda-kT+\tau)p^*(\lambda)\right]\, d\lambda \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_b[k]\int_{-\infty}^{\infty} p(\lambda-kT+\tau)p^*(\lambda)\, d\lambda
\end{aligned}
$$

PSD of a Linearly Modulated Signal

$$R_u(\tau) = \frac{1}{T} \sum_{k=-\infty}^{\infty} R_b[k] \int_{-\infty}^{\infty} p(\lambda - kT + \tau) p^*(\lambda)\, d\lambda$$

$$\int_{-\infty}^{\infty} p(\lambda + \tau) p^*(\lambda)\, d\lambda \quad \leftrightarrow \quad |P(f)|^2$$

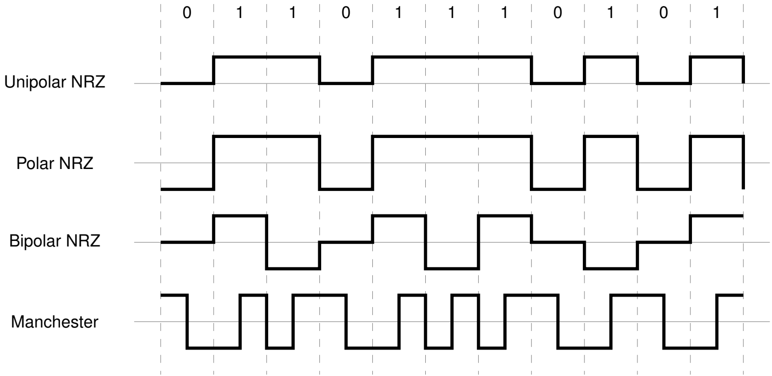
$$\int_{-\infty}^{\infty} p(\lambda - kT + \tau) p^*(\lambda)\, d\lambda \quad \leftrightarrow \quad |P(f)|^2 e^{-j2\pi fkT}$$

$$\begin{aligned} S_u(f) = \mathcal{F}[R_u(\tau)] \quad &= \quad \frac{|P(f)|^2}{T} \sum_{k=-\infty}^{\infty} R_b[k] e^{-j2\pi fkT} \\ &= \quad S_b\left(e^{j2\pi fT}\right) \frac{|P(f)|^2}{T} \end{aligned}$$

where $S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k] z^{-k}$.

PSD of Line Codes

Line Codes



Further reading: *Digital Communications*, Simon Haykin, Chapter 6

Unipolar NRZ

- Symbols independent and equally likely to be 0 or A

$$P\left(b[n]=0\right)=P\left(b[n]=A\right)=\frac{1}{2}$$

- Autocorrelation of $b[n]$ sequence

$$R_b[k]=\begin{cases}\frac{A^2}{2} & k=0\\ \frac{A^2}{4} & k\neq 0\end{cases}$$

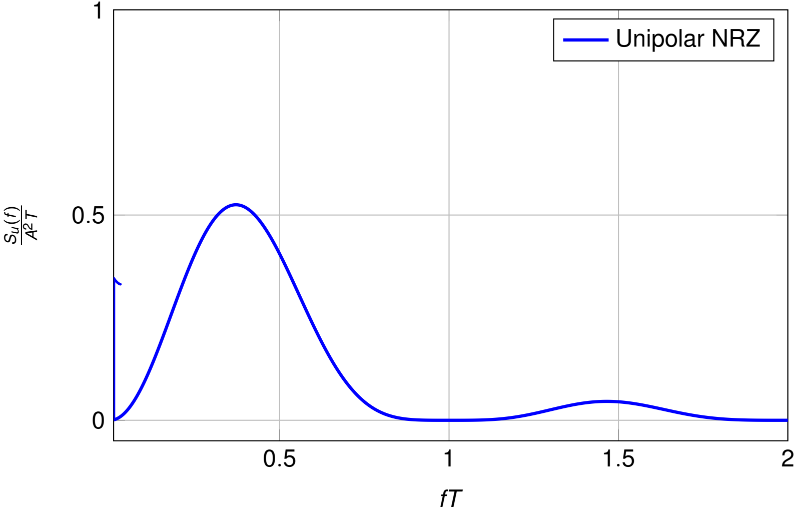
- $p(t)=I_{[0,T)}(t)\Rightarrow P(f)=T\operatorname{sinc}(fT)e^{-j\pi fT}$
- Power Spectral Density

$$S_u(f)=\frac{|P(f)|^2}{T}\sum_{k=-\infty}^{\infty}R_b[k]e^{-j2\pi kfT}$$

Unipolar NRZ

$$
\begin{aligned}
S_u(f) &= \frac{A^2 T}{4}\text{sinc}^2(fT) + \frac{A^2 T}{4}\text{sinc}^2(fT)\sum_{k=-\infty}^{\infty} e^{-j2\pi kfT} \\
&= \frac{A^2 T}{4}\text{sinc}^2(fT) + \frac{A^2}{4}\text{sinc}^2(fT)\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \\
&= \frac{A^2 T}{4}\text{sinc}^2(fT) + \frac{A^2}{4}\delta(f)
\end{aligned}
$$

Normalized PSD plot



Polar NRZ

- Symbols independent and equally likely to be $-A$ or A

$$P\left(b[n]=-A\right)=P\left(b[n]=A\right)=\frac{1}{2}$$

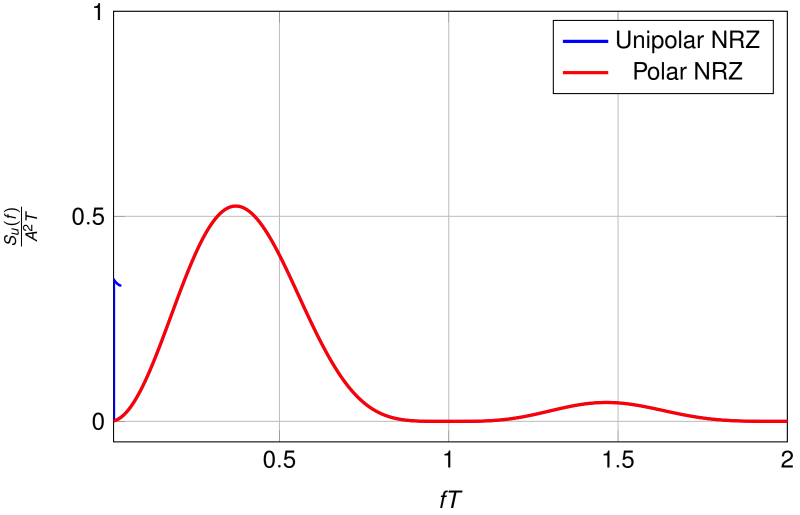
- Autocorrelation of $b[n]$ sequence

$$R_b[k]=\begin{cases} A^2 & k=0 \\ 0 & k\neq 0 \end{cases}$$

- $P(f)=T\text{sinc}(fT)e^{-j\pi fT}$
- Power Spectral Density

$$S_u(f)=A^2T\text{sinc}^2(fT)$$

Normalized PSD plots



Manchester

- Symbols independent and equally likely to be $-A$ or A

$$P\left(b[n]=-A\right)=P\left(b[n]=A\right)=\frac{1}{2}$$

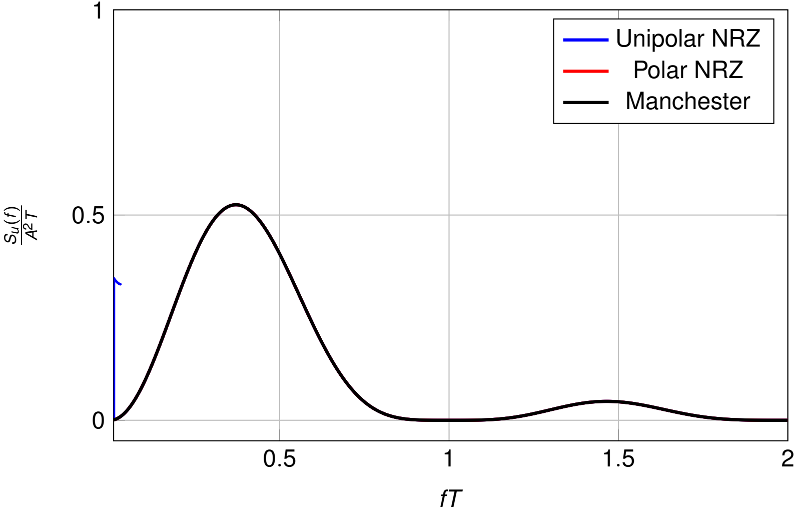
- Autocorrelation of $b[n]$ sequence

$$R_b[k]=\begin{cases} A^2 & k=0 \\ 0 & k\neq 0 \end{cases}$$

- $P(f)=jT\text{sinc}\left(\frac{fT}{2}\right)\sin\left(\frac{\pi fT}{2}\right)e^{-j\pi fT}$
- Power Spectral Density

$$S_u(f)=A^2T\text{sinc}^2\left(\frac{fT}{2}\right)\sin^2\left(\frac{\pi fT}{2}\right)$$

Normalized PSD plots



Bipolar NRZ

- Successive 1's have alternating polarity

$$\begin{aligned} 0 &\rightarrow \text{Zero amplitude} \\ 1 &\rightarrow +A \text{ or } -A \end{aligned}$$

- Probability mass function of $b[n]$

$$\begin{aligned} P\left(b[n]=0\right) &= \frac{1}{2} \\ P\left(b[n]=-A\right) &= \frac{1}{4} \\ P\left(b[n]=A\right) &= \frac{1}{4} \end{aligned}$$

- Symbols are identically distributed but they are not independent

Bipolar NRZ

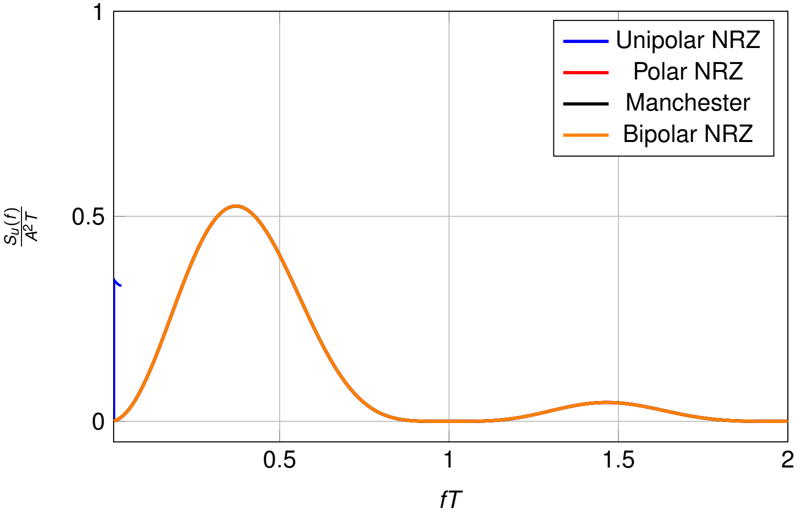
- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} A^2/2 & k = 0 \\ -A^2/4 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

- Power Spectral Density

$$\begin{aligned} S_u(f) &= T\text{sinc}^2(fT)\left[\frac{A^2}{2} - \frac{A^2}{4}\left(e^{j2\pi fT} + e^{-j2\pi fT}\right)\right] \\ &= \frac{A^2 T}{2}\text{sinc}^2(fT)\left[1 - \cos(2\pi fT)\right] \\ &= A^2 T\text{sinc}^2(fT)\sin^2(\pi fT) \end{aligned}$$

Normalized PSD plots



PSD of Passband Modulated Signals

Relating the PSDs of a Passband Modulated Signal and its Complex Envelope

- Definitions
  - $s_p(t)$ is a passband signal realization with complex envelope $s(t)$
  - For observation interval T_o, $\hat{s}_p(t) = s_p(t) I_{\left[-\frac{T_o}{2}, \frac{T_o}{2}\right]}(t)$
  - $\hat{s}_p(t)$ has complex envelope $\hat{s}(t)$
  - $\hat{s}_p(t) \leftrightarrow \hat{S}_p(f)$ and $\hat{s}(t) \leftrightarrow \hat{S}(f)$
- PSD approximations for $s_p(t)$ and $s(t)$

$$S_{s_p}(f) \approx \frac{\left|\hat{S}_p(f)\right|^2}{T_o}, \quad S_s(f) \approx \frac{\left|\hat{S}(f)\right|^2}{T_o}$$

- From the relationship between the deterministic signals

$$\hat{S}_p(f) = \frac{1}{\sqrt{2}} \left(\hat{S}(f - f_c) + \hat{S}^*(-f - f_c) \right)$$

- Since $\hat{S}(f - f_c)$ and $\hat{S}^*(-f - f_c)$ do not overlap, we have

$$\left|\hat{S}_p(f)\right|^2 = \frac{1}{2} \left(\left|\hat{S}(f - f_c)\right|^2 + \left|\hat{S}^*(-f - f_c)\right|^2 \right)$$

Relating the PSDs of a Passband Modulated Signal and its Complex Envelope

- Dividing by T_o

$$\frac{\left|\hat{S}_p(f)\right|^2}{T_o} = \frac{1}{2}\left(\frac{\left|\hat{S}(f-f_c)\right|^2}{T_o} + \frac{\left|\hat{S}^*(-f-f_c)\right|^2}{T_o}\right)$$

- As the observation interval $T_o \to \infty$, we get

$$S_{s_p}(f) = \frac{1}{2}\left[S_s(f-f_c) + S_s(-f-f_c)\right]$$

- By a similar argument, we get

$$S_s(f) = 2S_{s_p}(f+f_c)u(f+f_c)$$

References

- Section 2.5, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008
- Section 2.3.1, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008
- Chapter 6, *Digital Communications*, Simon Haykin, 2006