EE 703: Digital Message Transmission (Autumn 2021)
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Midsem Exam: 20 points — Date: September 11, 2021

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1. [5 points] Let $p(t)$ be a signal which is nonzero in the interval $[0, T]$ with Fourier transform $P(f)$. Suppose $u(t)$ is a polar NRZ signal given by

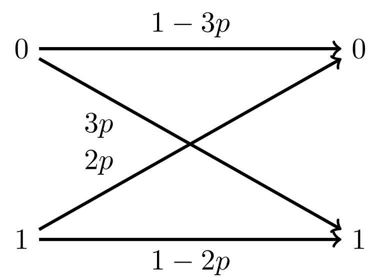
$$u(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

where the symbols b_n are independent and identically distributed with probability mass function $\Pr[b_n = A] = \Pr[b_n = -A] = \frac{1}{2}$ for some real value $A > 0$.

Suppose $v(t) = u(t) + u(t - T)$. Derive the power spectral density of $v(t)$. Simplify your answer such that it does not contain any infinite summations. *Hint: The formula for the power spectral density of of a line code signal $u(t)$ is given by*

$$S_u(f) = \frac{|P(f)|^2}{T} \sum_{k=-\infty}^{\infty} R_b[k] e^{-j2\pi k f T}.$$

2. [5 points] Suppose the input to the following binary channel is equally likely to be 0 or 1. Based on the output of the channel, we would like to make a decision about the input bit.



Assuming $0 \leq p < \frac{1}{3}$, derive the following:

(a) The optimal decision rule which minimizes the probability of decision error.

(b) The minimum probability of decision error of the optimal decision rule as a function of p.

3. [5 points] Consider the following ternary hypothesis testing problem where the hypotheses are equally likely.

$$\begin{aligned} H_1 &: Y \sim U[-1, 1], \\ H_2 &: Y \sim U[-2, 2], \\ H_3 &: Y \sim U[-3, 3], \end{aligned}$$

where $U[a, b]$ denotes the uniform distribution between real numbers a and b.

(a) Derive the optimal decision rule.

(b) Find the decision error probability of the optimal decision rule.

4. [5 points] Suppose X_1, X_2, X_3, X_4 are jointly Gaussian random variables. We are also given that X_1, X_2, X_3, X_4 are independent random variables.

   All four random variables have variance $\sigma^2 > 0$. X_1 and X_2 have mean $\mu > 0$ while X_3 and X_4 have mean 2μ.

   Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

$$H_1 \quad : \quad \mathbf{Y} = \begin{bmatrix} X_1 \\ X_2 + X_3 \\ X_4 \end{bmatrix},$$

$$H_2 \quad : \quad \mathbf{Y} = \begin{bmatrix} X_3 \\ X_1 + X_4 \\ X_2 \end{bmatrix}.$$

   (a) Derive the optimal decision rule. Show your steps and simplify the rule as much as possible.

   (b) Find the decision error probability of the optimal decision rule.