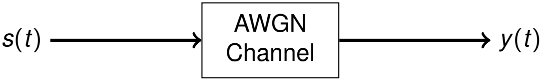
Optimal Receiver for the AWGN Channel

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

Additive White Gaussian Noise Channel



$$y(t) = s(t) + n(t)$$

- $s(t)$ Transmitted Signal
- $y(t)$ Received Signal
- $n(t)$ White Gaussian Noise

$$S_n(f) = \frac{N_0}{2} = \sigma^2$$

$$R_n(\tau) = \sigma^2 \delta(\tau)$$

M-ary Signaling in AWGN Channel

- One of M continuous-time signals $s_1(t), \ldots, s_M(t)$ is sent
- The received signal is the transmitted signal corrupted by AWGN
- M hypotheses with prior probabilities π_i, $i = 1, \ldots, M$

$$
\begin{array}{ccc}
H_1 & : & y(t) = s_1(t) + n(t) \\
H_2 & : & y(t) = s_2(t) + n(t) \\
\vdots & & \vdots \\
H_M & : & y(t) = s_M(t) + n(t)
\end{array}
$$

- The model implicitly assumes that
  - the delay has been estimated and
  - there is no attenuation or other distortion.
- Random variables are easier to handle than random processes
- We derive an equivalent M-ary hypothesis testing problem involving only random vectors
- Two digressions follow
  - Gaussian random processes
  - Signal space representation

Gaussian Random Processes

Gaussian Random Process

Definition

A random process $X(t)$ is Gaussian if its samples $X(t_1), \ldots, X(t_n)$ are jointly Gaussian for any $n \in \mathbb{N}$ and distinct sample locations $t_1, t_2, \ldots, t_n$.

Let $\mathbf{X} = \begin{bmatrix} X(t_1) & \cdots & X(t_n) \end{bmatrix}^T$ be the vector of samples. The joint density is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m}) \right)$$

where

$$\mathbf{m} = E[\mathbf{X}], \;\; \mathbf{C} = E\left[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T \right]$$

Properties

- The mean and autocorrelation functions completely characterize a Gaussian random process.
- Wide-sense stationary Gaussian processes are strictly stationary.

White Gaussian Noise

Definition

A zero mean WSS Gaussian random process with power spectral density

$$S_n(f) = \frac{N_0}{2}.$$

$\frac{N_0}{2}$ is termed the two-sided PSD and has units Watts per Hertz.

Remarks

- Autocorrelation function $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$
- Infinite Power! Ideal model of Gaussian noise occupying more bandwidth than the signals of interest.

White Gaussian Noise through Correlators

- Consider the output of a correlator with WGN input

$$Z = \int_{-\infty}^{\infty} n(t)u(t)\, dt = \langle n, u \rangle$$

  where $u(t)$ is a deterministic finite-energy real signal
- Z is a Gaussian random variable
- The mean of Z is

$$E[Z] = \int_{-\infty}^{\infty} E\left[n(t)\right] u(t)\, dt = 0$$

- The variance of Z is

$$\begin{aligned} \mathrm{var}(Z) &= E\left[(\langle n, u \rangle)^2\right] = E\left[\int_{-\infty}^{\infty} n(t)u(t)\, dt \int_{-\infty}^{\infty} n(s)u(s)\, ds\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t)u(s)E\left[n(t)n(s)\right]\ dt\, ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t)u(s)\frac{N_0}{2}\delta(t-s)\, dt\, ds \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty} u^2(t)\, dt = \frac{N_0}{2}\|u\|^2 \end{aligned}$$

White Gaussian Noise through Correlators

Proposition

Let $u_1(t)$ and $u_2(t)$ be finite-energy real signals and let $n(t)$ be WGN with PSD $S_n(f) = \frac{N_0}{2}$. Then $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian with covariance

$$\operatorname{cov}(\langle n, u_1 \rangle, \langle n, u_2 \rangle) = \frac{N_0}{2} \langle u_1, u_2 \rangle.$$

Proof

To prove that $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian, consider a linear combination $a\langle n, u_1 \rangle + b\langle n, u_2 \rangle$

$$a\langle n, u_1 \rangle + b\langle n, u_2 \rangle = \int_{-\infty}^{\infty} n(t) \left[a u_1(t) + b u_2(t) \right] \, dt.$$

This is the result of passing $n(t)$ through a correlator. So it is a Gaussian random variable.

White Gaussian Noise through Correlators

Proof (continued)

$$
\begin{aligned}
\text{cov}\left(\langle n, u_1\rangle, \langle n, u_2\rangle\right) &= E\left[\langle n, u_1\rangle\langle n, u_2\rangle\right] \\
&= E\left[\int_{-\infty}^{\infty} n(t)u_1(t)\,dt \int_{-\infty}^{\infty} n(s)u_2(s)\,ds\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1(t)u_2(s)E\left[n(t)n(s)\right]\,dt\,ds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1(t)u_2(s)\frac{N_0}{2}\delta(t-s)\,dt\,ds \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} u_1(t)u_2(t)\,dt \\
&= \frac{N_0}{2}\langle u_1, u_2\rangle
\end{aligned}
$$

If $u_1(t)$ and $u_2(t)$ are orthogonal, $\langle n, u_1\rangle$ and $\langle n, u_2\rangle$ are independent.

Signal Space Representation

Signal Space Representation of Waveforms

- Given M finite energy waveforms, construct an orthonormal basis

$$s_1(t), \ldots, s_M(t) \rightarrow \underbrace{\phi_1(t), \ldots, \phi_N(t)}_{\text{Orthonormal basis}}$$

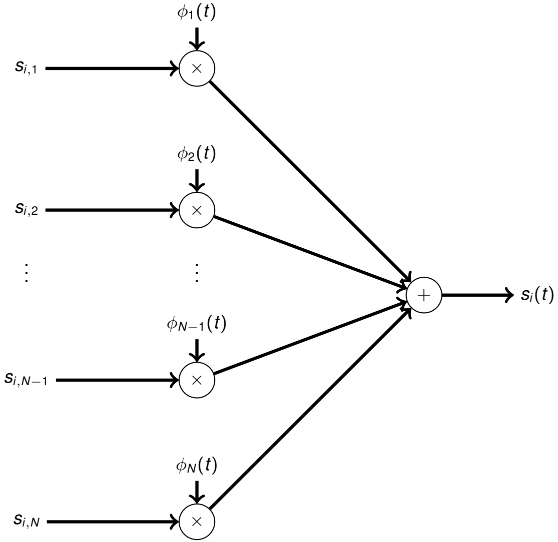
$$\langle \phi_i, \phi_j \rangle = \int_{-\infty}^{\infty} \phi_i(t)\phi_j^*(t)\, dt = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Each $s_i(t)$ is a linear combination of the basis vectors

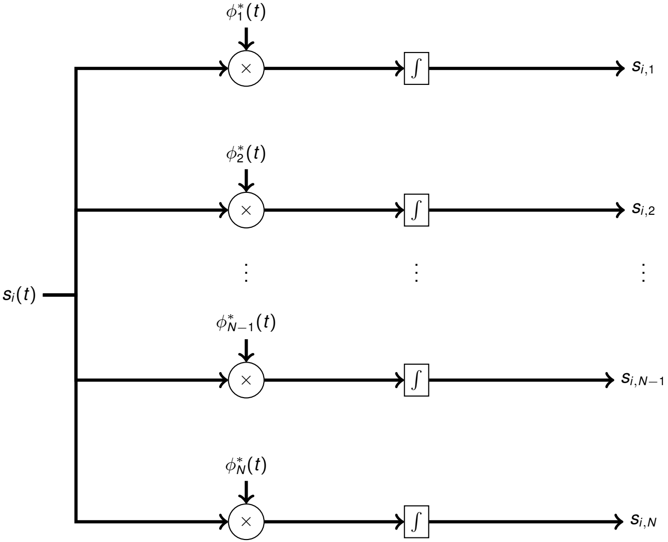
$$s_i(t) = \sum_{n=1}^{N} s_{i,n}\phi_n(t), \qquad i = 1, \ldots, M$$

- $s_i(t)$ is represented by the vector $\mathbf{s}_i = \begin{bmatrix} s_{i,1} & \cdots & s_{i,N} \end{bmatrix}^T$
- The set $\{\mathbf{s}_i : 1 \leq i \leq M\}$ is called the signal space representation or constellation

Constellation Point to Waveform



Waveform to Constellation Point



Gram-Schmidt Orthogonalization Procedure

- Algorithm for calculating orthonormal basis for $s_1(t), \ldots, s_M(t)$
- Consider $M = 1$

$$\phi_1(t) = \frac{s_1(t)}{\|s_1\|}$$

  where $\|s_1\|^2 = \langle s_1, s_1 \rangle$
- Consider $M = 2$

$$\phi_1(t) = \frac{s_1(t)}{\|s_1\|}, \quad \phi_2(t) = \frac{\gamma(t)}{\|\gamma\|}$$

  where $\gamma(t) = s_2(t) - \langle s_2, \phi_1 \rangle \phi_1(t)$
- Consider $M = 3$

$$\phi_1(t) = \frac{s_1(t)}{\|s_1\|}, \quad \phi_2(t) = \frac{\gamma_1(t)}{\|\gamma_1\|}, \quad \phi_3(t) = \frac{\gamma_2(t)}{\|\gamma_2\|}$$

  where

$$\begin{aligned} \gamma_1(t) &= s_2(t) - \langle s_2, \phi_1 \rangle \phi_1(t) \\ \gamma_2(t) &= s_3(t) - \langle s_3, \phi_1 \rangle \phi_1(t) - \langle s_3, \phi_2 \rangle \phi_2(t) \end{aligned}$$

Gram-Schmidt Orthogonalization Procedure

- In general, given $s_1(t), \dots, s_M(t)$ the kth basis function is

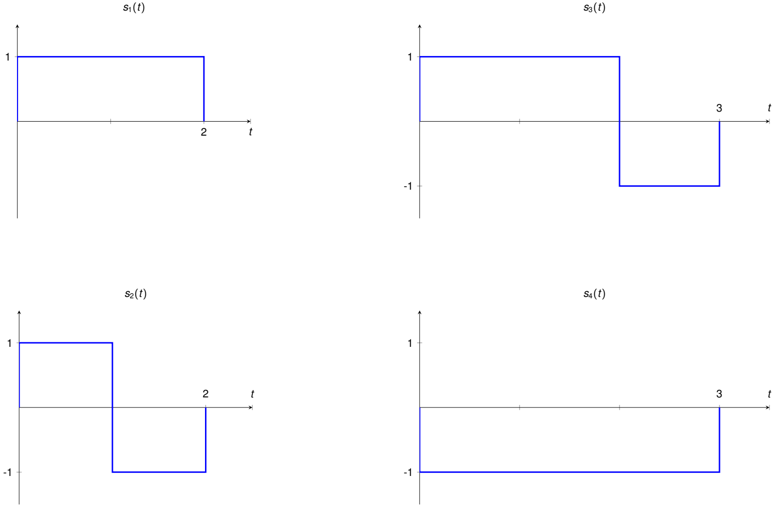
$$\phi_k(t) = \frac{\gamma_k(t)}{\|\gamma_k\|}$$

  where

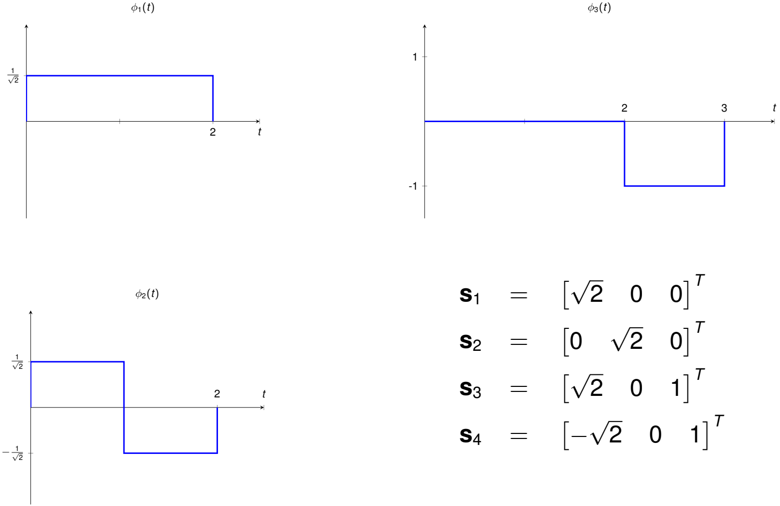
$$\gamma_k(t) = s_k(t) - \sum_{i=1}^{k-1} \langle s_k, \phi_i \rangle \phi_i(t)$$

  is not the zero function
- If $\gamma_k(t)$ is zero, $s_k(t)$ is a linear combination of $\phi_1(t), \dots, \phi_{k-1}(t)$. It does not contribute to the basis.

Gram-Schmidt Procedure Example



Gram-Schmidt Procedure Example



$$
\begin{aligned}
\mathbf{s}_1 &= \begin{bmatrix} \sqrt{2} & 0 & 0 \end{bmatrix}^T \\
\mathbf{s}_2 &= \begin{bmatrix} 0 & \sqrt{2} & 0 \end{bmatrix}^T \\
\mathbf{s}_3 &= \begin{bmatrix} \sqrt{2} & 0 & 1 \end{bmatrix}^T \\
\mathbf{s}_4 &= \begin{bmatrix} -\sqrt{2} & 0 & 1 \end{bmatrix}^T
\end{aligned}
$$

Properties of Signal Space Representation

- Energy

$$E_m = \int_{-\infty}^{\infty} |s_m(t)|^2 \, dt = \sum_{n=1}^{N} |s_{m,n}|^2 = \|\mathbf{s}_m\|^2$$

- Inner product

$$\langle s_i(t), s_j(t) \rangle = \langle \mathbf{s}_i, \mathbf{s}_j \rangle$$

Optimal Receiver for the AWGN Channel

Restriction to Signal Space is Optimal

Theorem

For the M-ary hypothesis testing given by

$$\begin{aligned} H_1 &: \quad y(t) = s_1(t) + n(t) \\ \vdots & \qquad\qquad \vdots \\ H_M &: \quad y(t) = s_M(t) + n(t) \end{aligned}$$

there is no loss in detection performance by using the optimal decision rule for the following M-ary hypothesis testing problem

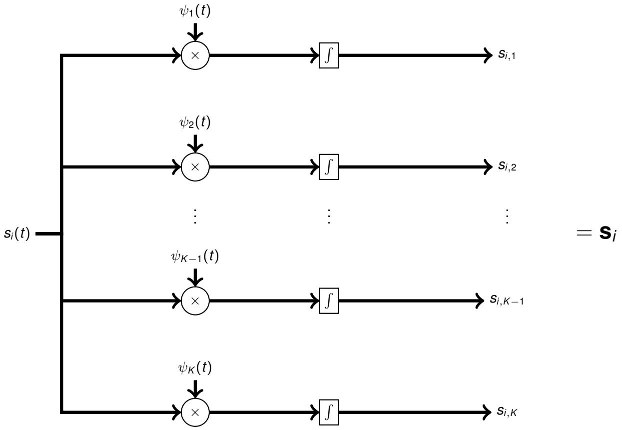
$$\begin{aligned} H_1 &: \quad \mathbf{Y} = \mathbf{s}_1 + \mathbf{N} \\ \vdots & \qquad\qquad \vdots \\ H_M &: \quad \mathbf{Y} = \mathbf{s}_M + \mathbf{N} \end{aligned}$$

where $\mathbf{Y}$, $\mathbf{s}_i$ *and* $\mathbf{N}$ *are the projections of* $y(t)$, $s_i(t)$ *and* $n(t)$ *respectively onto the signal space spanned by* $\{s_i(t)\}$.

Projection of Signals onto Signal Space

- Consider an orthonormal basis $\{\psi_i(t) \mid i = 1, \dots, K\}$ for the space spanned by $\{s_i(t) \mid i = 1, \dots, M\}$
- Projection of $s_i(t)$ onto the signal space is

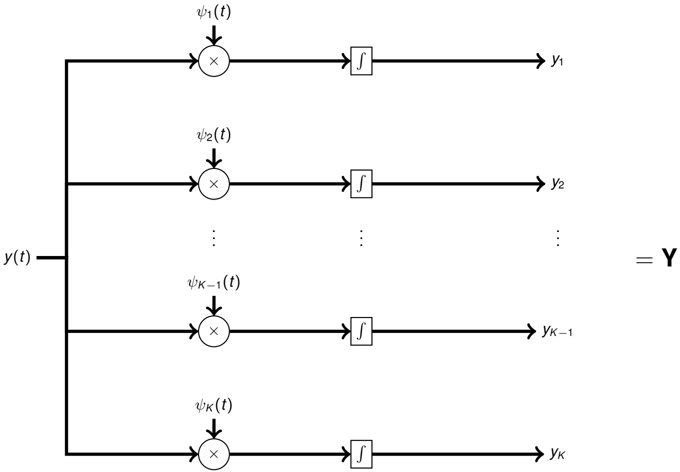
$$\mathbf{s}_i = \begin{bmatrix} \langle s_i, \psi_1 \rangle & \cdots & \langle s_i, \psi_K \rangle \end{bmatrix}^T$$



Projection of Observed Signal onto Signal Space

- Projection of $y(t)$ onto the signal space is

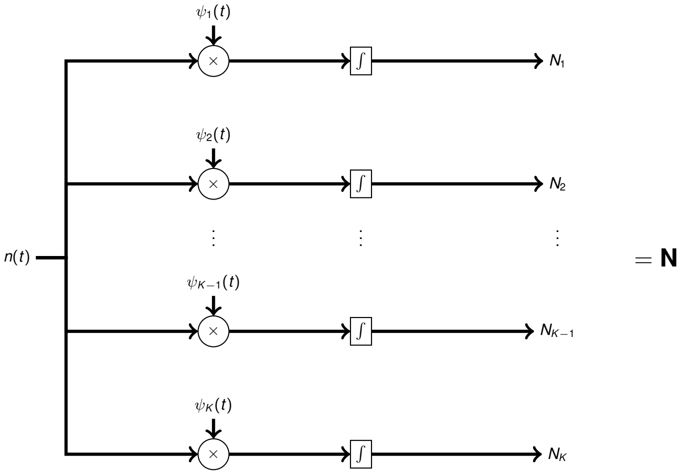
$$\mathbf{Y} = \begin{bmatrix} \langle y, \psi_1 \rangle & \cdots & \langle y, \psi_K \rangle \end{bmatrix}^T$$



Projection of Noise onto Signal Space

- Projection of $n(t)$ onto the signal space is

$$\mathbf{N} = \begin{bmatrix} \langle n, \psi_1 \rangle & \cdots & \langle n, \psi_K \rangle \end{bmatrix}^T$$



Proof of Theorem

- $\mathbf{Y} = \begin{bmatrix} \langle y, \psi_1 \rangle & \cdots & \langle y, \psi_K \rangle \end{bmatrix}^T$
- Component of $y(t)$ orthogonal to the signal space is

$$y^{\perp}(t) = y(t) - \sum_{i=1}^{K} \langle y, \psi_i \rangle \psi_i(t)$$

- $y(t)$ is equivalent to $(\mathbf{Y}, y^{\perp}(t))$
- We claim that $y^{\perp}(t)$ is an irrelevant statistic

$$\begin{aligned} y^{\perp}(t) &= y(t) - \sum_{j=1}^{K} \langle y, \psi_j \rangle \psi_j(t) \\ &= s_i(t) + n(t) - \sum_{j=1}^{K} \langle s_i + n, \psi_j \rangle \psi_j(t) \\ &= n(t) - \sum_{j=1}^{K} \langle n, \psi_j \rangle \psi_j(t) = n^{\perp}(t) \end{aligned}$$

  where $n^{\perp}(t)$ is the component of $n(t)$ orthogonal to the signal space.
- $n^{\perp}(t)$ does not depend on which $s_i(t)$ was transmitted and is independent of $\mathbf{N}$, which makes $y^{\perp}(t)$ an irrelevant statistic.

M-ary Signaling in AWGN Channel

- M hypotheses with prior probabilities π_i, $i = 1, \ldots, M$

$$\begin{aligned} H_1 &: \quad \mathbf{Y} = \mathbf{s}_1 + \mathbf{N} \\ \vdots & \qquad\quad \vdots \\ H_M &: \quad \mathbf{Y} = \mathbf{s}_M + \mathbf{N} \end{aligned}$$

$$\begin{aligned} \mathbf{Y} &= \left[\langle y, \psi_1 \rangle \quad \cdots \quad \langle y, \psi_K \rangle \right]^T \\ \mathbf{s}_i &= \left[\langle s_i, \psi_1 \rangle \quad \cdots \quad \langle s_i, \psi_K \rangle \right]^T \\ \mathbf{N} &= \left[\langle n, \psi_1 \rangle \quad \cdots \quad \langle n, \psi_K \rangle \right]^T \end{aligned}$$

- $\mathbf{N} \sim N(\mathbf{m}, \mathbf{C})$ where $\mathbf{m} = \mathbf{0}$ and $\mathbf{C} = \sigma^2 \mathbf{I}$

$$\mathrm{cov}\left(\langle n, \psi_1 \rangle, \langle n, \psi_2 \rangle \right) = \sigma^2 \langle \psi_1, \psi_2 \rangle.$$

Optimal Receiver for the AWGN Channel

Theorem (MPE Decision Rule)

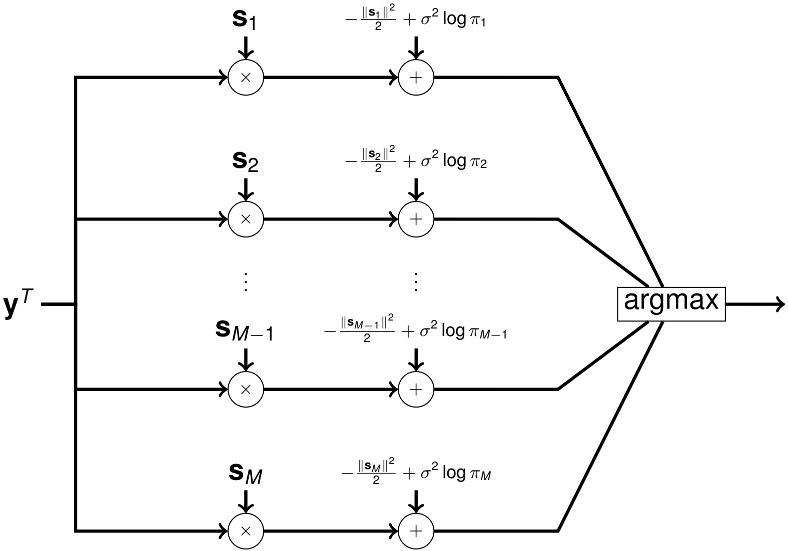
The MPE decision rule for M-ary signaling in AWGN channel is given by

$$\begin{aligned}\delta_{MPE}(\mathbf{y}) &= \underset{1 \le i \le M}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{s}_i\|^2 - 2\sigma^2 \log \pi_i \\ &= \underset{1 \le i \le M}{\operatorname{argmax}} \langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2} + \sigma^2 \log \pi_i\end{aligned}$$

Proof

$$\begin{aligned}\delta_{MPE}(\mathbf{y}) &= \underset{1 \le i \le M}{\operatorname{argmax}}\, \pi_i p_i(\mathbf{y}) \\ &= \underset{1 \le i \le M}{\operatorname{argmax}}\, \pi_i \exp\left(-\frac{\|\mathbf{y} - \mathbf{s}_i\|^2}{2\sigma^2}\right)\end{aligned}$$

MPE Decision Rule



Continuous-Time Version of MPE Rule

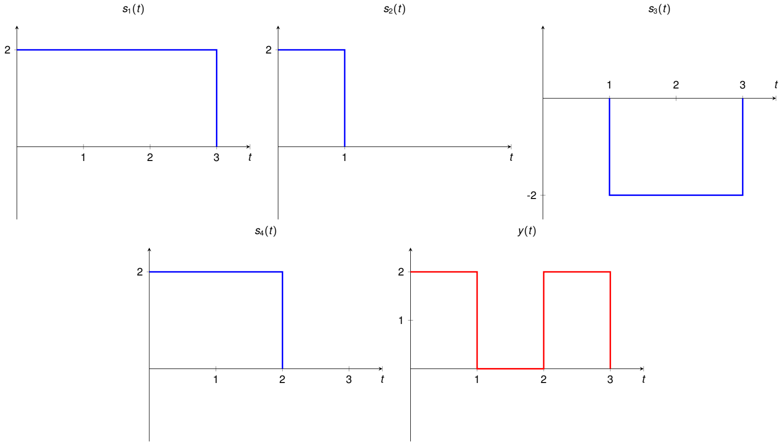
- Discrete-time version

$$\delta_{MPE}(\mathbf{y}) = \underset{1 \le i \le M}{\operatorname{argmax}} \langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2} + \sigma^2 \log \pi_i$$

- Continuous-time version

$$\delta_{MPE}(y) = \underset{1 \le i \le M}{\operatorname{argmax}} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2} + \sigma^2 \log \pi_i$$

MPE Decision Rule Example



Let $\pi_1 = \pi_2 = \dfrac{1}{3}, \pi_3 = \pi_4 = \dfrac{1}{6}, \sigma^2 = 1,$ and $\log 2 = 0.69$.

ML Receiver for the AWGN Channel

Theorem (ML Decision Rule)

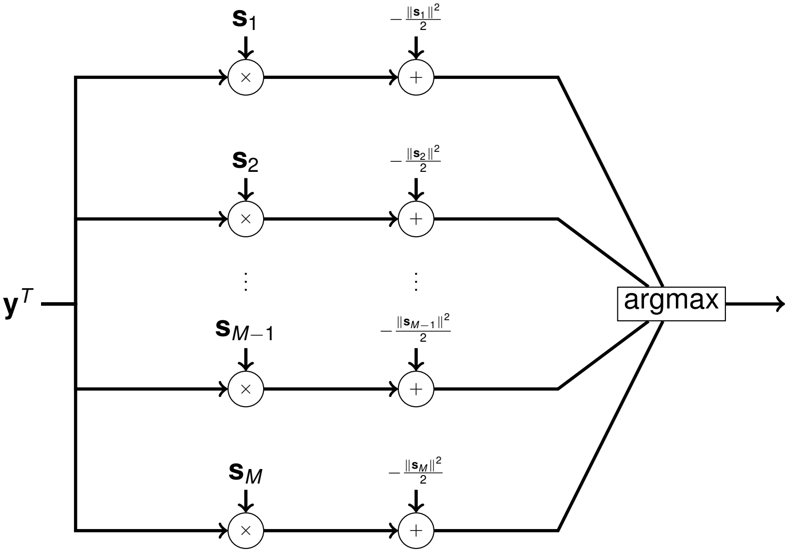
The ML decision rule for M-ary signaling in AWGN channel is given by

$$\begin{aligned}\delta_{ML}(\mathbf{y}) &= \underset{1\leq i\leq M}{\operatorname{argmin}} \|\mathbf{y}-\mathbf{s}_i\|^2 \\ &= \underset{1\leq i\leq M}{\operatorname{argmax}} \langle \mathbf{y}, \mathbf{s}_i\rangle - \frac{\|\mathbf{s}_i\|^2}{2}\end{aligned}$$

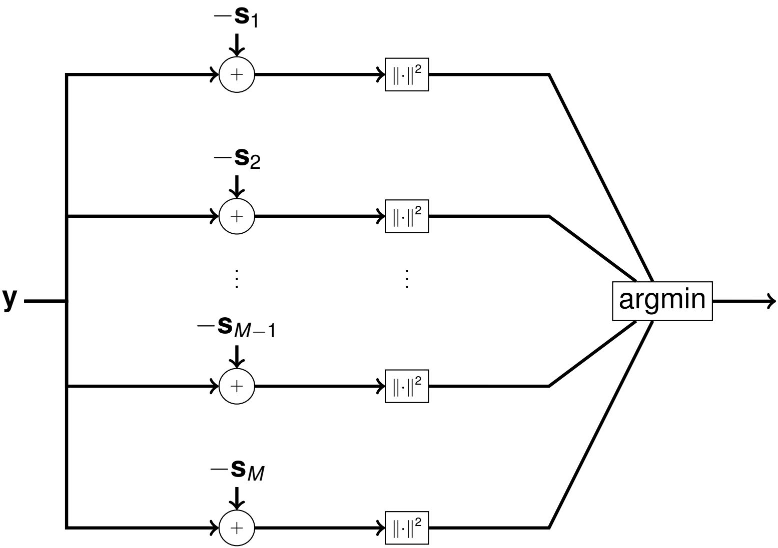
Proof

$$\begin{aligned}\delta_{ML}(\mathbf{y}) &= \underset{1\leq i\leq M}{\operatorname{argmax}}\, p_i(\mathbf{y}) \\ &= \underset{1\leq i\leq M}{\operatorname{argmax}} \exp\left(-\frac{\|\mathbf{y}-\mathbf{s}_i\|^2}{2\sigma^2}\right)\end{aligned}$$

ML Decision Rule



ML Decision Rule



Continuous-Time Version of ML Rule

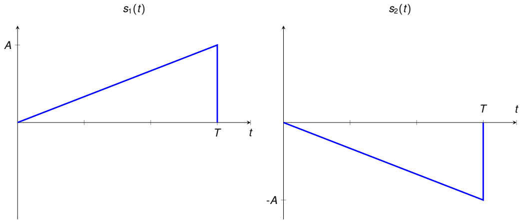
- Discrete-time version

$$\delta_{ML}(\mathbf{y}) = \underset{1 \le i \le M}{\text{argmax}} \langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2}$$

- Continuous-time version

$$\delta_{ML}(y) = \underset{1 \le i \le M}{\text{argmax}} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2}$$

ML Decision Rule for Antipodal Signaling



$$\delta_{ML}(y) = \underset{1 \le i \le 2}{\operatorname{argmax}} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2} = \underset{1 \le i \le 2}{\operatorname{argmax}} \langle y, s_i \rangle$$

$$\delta_{ML}(y) = 1 \iff \langle y, s_1 \rangle \ge \langle y, s_2 \rangle \iff \langle y, s_1 \rangle \ge 0$$

$$\langle y, s_1 \rangle = \int_0^T y(\tau) s_1(\tau) \, d\tau = (y \star s_{MF})(T)$$

where $s_{MF}(t) = s_1(T - t)$ is the matched filter.

References

- Sections 3.3, 3.4, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008