Preliminaries and Notation

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Complex Numbers

- A complex number z can be written as $z = x + jy$ where $x, y \in \mathbb{R}$ and $j = \sqrt{-1}$
    - We say $x = \mathrm{Re}(z)$ is the real part of z and
    - $y = \mathrm{Im}(z)$ is the imaginary part of z
- In polar form, $z = re^{j\theta}$ where

$$r = |z| = \sqrt{x^2 + y^2},$$
$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right).$$

- Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Inner Product

- Inner product of two $m \times 1$ complex vectors $\mathbf{s} = (s[1], \ldots, s[m])^T$ and $\mathbf{r} = (r[1], \ldots, r[m])^T$

$$\langle \mathbf{s}, \mathbf{r} \rangle = \sum_{i=1}^{m} s[i] r^*[i] = \mathbf{r}^H \mathbf{s}.$$

- Inner product of two complex-valued signals $s(t)$ and $r(t)$

$$\langle s, r \rangle = \int_{-\infty}^{\infty} s(t) r^*(t)\, dt$$

- Linearity properties

$$\begin{aligned} \langle a_1 s_1 + a_2 s_2, r \rangle &= a_1 \langle s_1, r \rangle + a_2 \langle s_2, r \rangle, \\ \langle s, a_1 r_1 + a_2 r_2 \rangle &= a_1^* \langle s, r_1 \rangle + a_2^* \langle s, r_2 \rangle. \end{aligned}$$

Energy and Cauchy-Schwarz Inequality

- Energy E_s of a signal s is defined as

$$E_s = \|s\|^2 = \langle s, s \rangle = \int_{-\infty}^{\infty} |s(t)|^2 \; dt$$

  where $\|s\|$ denotes the norm of s
- If energy of s is zero, then s must be zero "almost everywhere"
  - For our purposes, $\|s\| = 0 \implies s(t) = 0$ for all t
- Cauchy-Schwarz Inequality

$$|\langle s, r \rangle| \leq \|s\| \|r\|$$

  with equality $\iff$ for some complex constant a, $s(t) = ar(t)$

Convolution

- The convolution of two signals r and s is

$$q(t) = (s * r)(t) = \int_{-\infty}^{\infty} s(u) r(t-u)\, du$$

- The notation $s(t) * r(t)$ is also used to denote $(s * r)(t)$

Delta Function

- $\delta(t)$ is defined by the sifting property. For any finite energy signal $s(t)$

$$\int_{-\infty}^{\infty} s(t)\delta(t - t_0)\, dt = s(t_0)$$

- Convolution of a signal with a shifted delta function gives a shifted version of the signal

$$\delta(t - t_0) * s(t) = s(t - t_0)$$

- Sifting property also implies following properties
  - Unit area

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

  - Fourier transform

$$\mathcal{F}\left(\delta(t)\right) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft}\, dt = 1$$

Indicator Function and Sinc Function

- The indicator function of a set A is defined as

$$I_A(x) = \begin{cases} 1, & \text{for } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

- Sinc function

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x},$$

where the value at $x = 0$ is defined as 1

References

- pp 8 —13, Section 2.1, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008