Random Processes

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Random Process

Definition

An indexed collection of random variables $\{X(t) : t \in \mathcal{T}\}$.

Discrete-time Random Process $\mathcal{T} = \mathbb{Z}$ or $\mathbb{N}$

Continuous-time Random Process $\mathcal{T} = \mathbb{R}$

Statistics

Mean function

$$m_X(t) = E[X(t)]$$

Autocorrelation function

$$R_X(t_1, t_2) = E[X(t_1)X^*(t_2)]$$

Autocovariance function

$$C_X(t_1, t_2) = E\left[(X(t_1) - m_X(t_1))\,(X(t_2) - m_X(t_2))^*\right]$$

Stationary Random Process

Definition

A random process which is statistically indistinguishable from a delayed version of itself.

Properties

- For any $n \in \mathbb{N}$, $(t_1, \ldots, t_n) \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$, $(X(t_1), \ldots, X(t_n))$ has the same joint distribution as $(X(t_1 - \tau), \ldots, X(t_n - \tau))$.
- For $n = 1$, we have
$$F_{X(t)}(x) = F_{X(t+\tau)}(x)$$
for all t and τ. The first order distribution is independent of time.
- $m_X(t) = m_X(0)$
- For $n = 2$ and $\tau = t_2$, we have
$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(t_1-t_2),X(0)}(x_1, x_2)$$
for all t_1 and t_2. The second order distribution depends only on $t_1 - t_2$.
- $R_X(t_1, t_2) = R_X(t_1 - \tau, t_2 - \tau) = R_X(t_1 - t_2, 0)$

Wide Sense Stationary Random Process

Definition

A random process is WSS if

$$\begin{aligned} m_X(t) &= m_X(0) \quad \text{for all } t \text{ and} \\ R_X(t_1, t_2) &= R_X(t_1 - t_2, 0) \quad \text{for all } t_1, t_2. \end{aligned}$$

Autocorrelation function is expressed as a function of $\tau = t_1 - t_2$ as $R_X(\tau)$.

Definition (Power Spectral Density of a WSS Process)

The Fourier transform of the autocorrelation function.

$$S_X(f) = \mathcal{F}(R_X(\tau))$$

Energy Spectral Density of Signals

Definition

For a signal $x(t)$, the energy spectral density is defined as

$$E_x(f) = |X(f)|^2.$$

Motivation

Pass $x(t)$ through an ideal narrowband filter with response

$$H_{f_0}(f) = \begin{cases} 1, & \text{if } f_0 - \frac{\Delta f}{2} < f < f_0 + \frac{\Delta f}{2} \\ 0, & \text{otherwise} \end{cases}$$

Output is $Y(f) = X(f)H_{f_0}(f)$. Energy in output is given by

$$\int_{-\infty}^{\infty} |Y(f)|^2 \, df = \int_{f_0 - \frac{\Delta f}{2}}^{f_0 + \frac{\Delta f}{2}} |X(f)|^2 \, df \approx |X(f_0)|^2 \Delta f$$

Note

$$|X(f)|^2 \leftrightarrow x(t) \star x^*(-t) = \int_{-\infty}^{\infty} x(u)x^*(u-t) \, du$$

Power Spectral Density

Motivation

PSD characterizes spectral content of random signals which have infinite energy but finite power

Example (Finite-power infinite-energy signal)

Binary PAM signal

$$x(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

Time windowed realizations have finite energy

$$\begin{aligned} x_{T_o}(t) &= x(t) I_{[-\frac{T_o}{2}, \frac{T_o}{2}]}(t) \\ X_{T_o}(f) &= \mathcal{F}(x_{T_o}(t)) \\ \hat{S}_x(f) &= \frac{|X_{T_o}(f)|^2}{T_o} \quad \text{(PSD Estimate)} \end{aligned}$$

Definition (PSD of a realization)

$$\bar{S}_x(f) = \lim_{T_o \to \infty} \frac{|X_{T_o}(f)|^2}{T_o}$$

Autocorrelation Function of a Realization

Motivation

$$\hat{S}_x(f) = \frac{|X_{T_o}(f)|^2}{T_o} \quad \leftrightarrow \quad \frac{1}{T_o}\int_{-\infty}^{\infty} x_{T_o}(u)x_{T_o}^*(u-\tau)\,du$$

$$= \frac{1}{T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u)x_{T_o}^*(u-\tau)\,du$$

$$= \hat{R}_x(\tau) \qquad \text{(Autocorrelation Estimate)}$$

Definition (Autocorrelation function of a realization)

$$\bar{R}_x(\tau) = \lim_{T_o\to\infty} \frac{1}{T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u)x_{T_o}^*(u-\tau)\,du$$

The Two Definitions of Power Spectral Density

Definition (PSD of a WSS Process)

$$S_X(f) = \mathcal{F}\left(R_X(\tau)\right)$$

where $R_X(\tau) = E\left[X(t)X^*(t-\tau)\right]$.

Definition (PSD of a realization)

$$\bar{S}_x(f) = \mathcal{F}\left(\bar{R}_x(\tau)\right)$$

where

$$\bar{R}_x(\tau) = \lim_{T_o \to \infty} \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u) x_{T_o}^*(u-\tau)\, du$$

Both are equal for ergodic processes

Ergodic Process

Definition

A stationary random process is ergodic if time averages equal ensemble averages.

- Ergodic in mean

$$\lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt = E[X(t)]$$

- Ergodic in autocorrelation

$$\lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x^*(t-\tau)\, dt = R_X(\tau)$$

References

- Section 2.3, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008
- Page 15, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008