Indian Institute of Technology Bombay
Department of Electrical Engineering

Handout 13 EE 706 Communication Networks
Solutions to Assignment 2 February 15, 2010

1. Suppose a six-faced die is tossed and every face is equally likely to show up. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let X be the random variable which is equal to the number on the face which is pointing up.

   (a) What is the probability mass function (pmf) of X? [1 point]

   Ans.
   $\Pr[X = i] = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

   (b) What is the expected value $E(X)$ of X? [1 point]

   Ans.
   $E[X] = \sum_{i=1}^{6} i \Pr[X = i] = 3.5$

   (c) The variance of a random variable Y is defined as $E\left[(Y - E[Y])^2\right]$. What is the variance of X? [2 points]

   Ans.
   $E\left[(X - E[X])^2\right] = \sum_{i=1}^{6} (i - 3.5)^2 \Pr[X = i] = \frac{35}{12}$

2. A *Bernoulli* random variable X has pmf

$$
\begin{aligned}
p(0) &= P[X = 0] = 1 - q \\
p(1) &= P[X = 1] = q
\end{aligned}
$$

   where $0 \leq q \leq 1$. What is the mean (expected value) and variance of X?

   [2 points]

   Ans.
   Mean of X is $E[X] = 0 \times (1 - q) + 1 \times q = q$. Variance of X is $E[(X - q)^2] = (0 - q)^2 \times (1 - q) + (1 - q)^2 \times q = q^2(1 - q) + q(1 - q)^2 = q(1 - q)$.

3. A *binomial* random variable is defined as the number of successes in n independent experiments, each of which results in a success with probability q and in a failure with probability $1 - q$.

   (a) Argue that the number of sequences of n outcomes with i successes and $n - i$ failures is

$$
\frac{n(n-1)\cdots(n-i+1)}{i(i-1)(i-2)\cdots 1}.
$$

   [1 point]

   Ans.

Given n positions, a success can be placed in any one of them. The next sucess can then be placed in the remaining $n-1$ positions. Continuing in this manner we get the number of ways of placing i *unique* successes is $n(n-1)(n-2)\ldots(n-i+1)$. But since the successes are indistinguishable, $i!$ placements which are different when the successes are distinguishable become the same. So the number of ways in which i indistinguishable successes can be placed in n positions is

$$\frac{n(n-1)(n-2)\cdots(n-i+1)}{i(i-1)(i-2)\cdots 1} = \binom{n}{i}.$$

(b) Using the above argument, derive the pmf of a binomial random variable. [1 point]

Ans.
$\Pr[X = i] = \binom{n}{i} q^i (1-q)^{n-i}$.

(c) Derive the expected value of the binomial random variable in the following two ways.

i. Directly from the pmf. [2 points]
Ans.
$E[X] = \sum_{i=0}^{n} i \binom{n}{i} q^i (1-q)^{n-i} = (1-q)^n \sum_{i=0}^{n} i \binom{n}{i} \left(\frac{q}{1-q}\right)^i$. By the binomial theorem, $\sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n$ and by differentiation we get $\sum_{i=0}^{n} i \binom{n}{i} x^i = nx(1+x)^{n-1}$. By setting $x = \frac{q}{1-q}$, we get $E[X] = nq$.

ii. Writing the binomial random variable as a sum of *some other* random variables and using the linearity property of expectation, $E[X+Y] = E[X] + E[Y]$ where X and Y are random variables. [2 points]
Ans.
A binomial random variable X can be expressed as a sum of n Bernoulli random variables Y_i $(i = 1, 2, 3, \ldots, n)$ where each Y_i takes value 0 with probability $1-q$ and 1 with probability q. Since $E[Y_i] = q$, $E[X] = E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} E[Y_i] = nq$.

4. The probability mass function of a geometric random variable having parameter q is given by

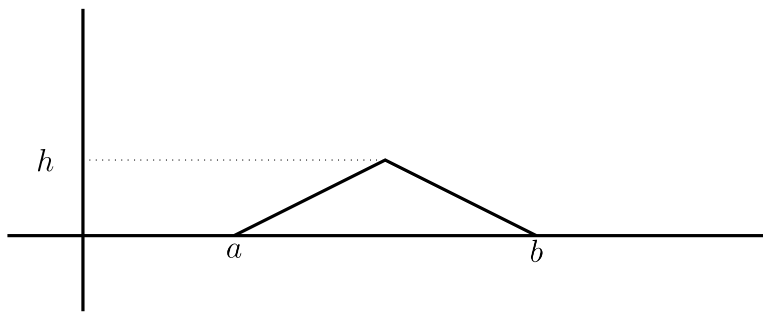
$$p(n) = P[X = n] = (1-q)^{n-1} q, \quad n = 1, 2, \ldots$$

Derive its mean (expected value) and variance. [4 points]

Ans.

Using the fact that $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$, we get $E[X] = \sum_{n=1}^{\infty} n(1-q)^{n-1} q = \frac{q}{[1-(1-q)]^2} = \frac{1}{q}$. Using the fact that $\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}$, the variance can be shown to be $\frac{1-q}{q^2}$.

5. A continous random variable has probability density function (pdf) as shown in the above figure. The dotted line is not part of the pdf. It indicates that the height of the triangle is h.



(a) If the triangle is isosceles, what should the value of h be in terms of a and b for the function plotted in the figure to be a valid pdf? [2 points]

Ans.

For the function to be valid pdf, the area under it should be 1. So $\frac{1}{2}h(b-a) = 1 \Longrightarrow h = \frac{2}{b-a}$.

(b) What is the expected value of this random variable if $a = 0$ and $b = 2$? [2 points]

Ans.

For $a = 0$ and $b = 2$, the pdf is

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2-x & \text{for } 1 < x \leq 2 \end{cases}$$

So $E[X] = \int_0 2xf(x)\ dx = \int_0^1 x^2\ dx + \int_1^2 x(2-x)\ dx = 1$

6. Let the generator polynomial $g(X) = X^4 + X^3 + X^2 + 1$ be used to generate CRC check bits. Without doing actual division, argue why each of the following error patterns can be detected by the CRC error detection procedure. In each case, the number of the transmitted bits is equal to the number of bits in the error pattern. [10 points]

Ans. An error pattern can be detected if the corresponding polynomial is not divisible by $g(X)$ or in other words if the corresponding polynomial is not a multiple of $g(X)$.

(a) 0100000000

The error polynomial corresponding to this error pattern is X^8. This is not a multiple of $g(X)$ because any multiple will have at least two terms.

(b) 0100100

The error polynomial corresponding to this error pattern is $X^5 + X^2 = X^2(X^3 + 1)$. Now $g(X)$ does not divide X^2 because it has only one term. Also $g(X)$ does not divide $X^3 + 1$ because its degree is less than the degree of $g(X)$. The other way to argue this is to say $g(X) = (X+1)(X^3 + X + 1)$ where $X^3 + X + 1$ is a primitive polynomial (you will have to show that it is primitive). Hence the smallest value of m for which $X^3 + X + 1$ divides $X^m + 1$ is $2^3 - 1 = 7$ which is greater than 3. So $g(X)$ does not divide $X^3 + 1$. The latter argument works even if the error pattern was $X^2(X^5 + 1)$.

(c) 1010000000001101

Since $g(X) = (X+1)(X^3+X+1)$, $g(1) = 0$. Then if $e(X)$ is a multiple of $g(X)$, $e(1) = 0$. But the error pattern has an odd number of errors and hence the error polynomial has an odd number of terms. So $e(1) = 1$ and so $e(X)$ cannot be a multiple of $g(X)$.

(d) 00000000111100000000

Ans. The error polynomial corresponding to this error pattern is $e(X) = X^{11} + X^{10} + X^9 + X^8 = X^8(X^3 + X^2 + X + 1)$. Now $g(X)$ does not divide X^8 because it has only term. Also $g(X)$ does not divide $X^3 + X^2 + X + 1$ because its degree is less than the degree of $g(X)$.