EE 720: An Introduction to Number Theory and Cryptography (Spring 2018)

Lecture 9 — February 7, 2018

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1 Lecture Plan

- Recap the definition of CPA-security
- Define pseudorandom functions
- Give construction of CPA-secure encryption scheme

2 Recap

Chosen-Plaintext Attacks and CPA-Security

- Consider the following experiment $\texttt{PrivK}^{\texttt{cpa}}_{\mathcal{A},\Pi}(n)$:
  1. A key k is generated by running $\texttt{Gen}(1^n)$.
  2. The adversary $\mathcal{A}$ is given 1^n and oracle access to $\texttt{Enc}_k(\cdot)$, and outputs a pair of messages $m_0, m_1 \in \mathcal{M}$ with $|m_0| = |m_1|$.
  3. A uniform bit $b \in \{0, 1\}$ is chosen. Ciphertext $c \leftarrow \texttt{Enc}_k(m_b)$ is computed and given to $\mathcal{A}$.
  4. The adversary $\mathcal{A}$ continues to have oracle access to $\texttt{Enc}_k(\cdot)$, and outputs a bit b'.
  5. The output of the experiment is defined to be 1 if $b' = b$, and 0 otherwise. If output is 1, we say that $\mathcal{A}$ succeeds.

Definition. *A private-key encryption scheme $\Pi = (\texttt{Gen}, \texttt{Enc}, \texttt{Dec})$ has* ***indistinguishable encryptions under a plaintext attack****, or is* ***CPA-secure****, if for all probabilistic polynomial-time adversaries $\mathcal{A}$ there is a negligible function $\texttt{negl}$ such that, for all n,*

$$\Pr\left[\texttt{PrivK}^{\texttt{cpa}}_{\mathcal{A},\Pi}(n) = 1\right] \leq \frac{1}{2} + \texttt{negl}(n).$$

- Note that no deterministic encryption scheme can be CPA-secure.

3 Pseudorandom Functions

- Pseudorandom functions are "random-looking" functions.
- In this case, pseudorandomness will be a property of a distribution over functions.

- Given a security parameter n, a keyed function $F : \{0,1\}^{l_{key}(n)} \times \{0,1\}^{l_{in}(n)} \to \{0,1\}^{l_{out}(n)}$ is a two-input function, where the first input is called the key and is denoted by k. The functions l_{key}, l_{in}, l_{out} specify the lengths of the key, second input, and output respectively.

- We will only consider *efficient* keyed functions, i.e. there is a polynomial-time algorithm that computes $F(k,x)$ given k and x.

- If the key k is fixed, we get a single-input function $F_k : \{0,1\}^{l_{in}(n)} \to \{0,1\}^{l_{out}(n)}$ defined by $F_k(x) = F(k,x)$.

- F is said to be *length-preserving* when $l_{key}(n) = l_{in}(n) = l_{out}(n) = n$.

- For simplicity, let us assume that F is length-preserving.

- Let $\texttt{Func}_n$ be the set of all functions with domain and range equal to $\{0,1\}^n$.

- Informally, a keyed function F is said to be *pseudorandom* if the function F_k (for a uniform key k) is indistinguishable from a function chosen uniformly from $\texttt{Func}_n$. No efficient adversary should be able to distinguish (with a success probability non-negligibly better than $\frac{1}{2}$) whether it is interacting with F_k (for uniform k) or f (where f is uniformly chosen from $\texttt{Func}_n$).

- Note that $|\texttt{Func}_n| = 2^{n \cdot 2^n}$. Visualize a lookup table having 2^n rows with each row containing an n-bit string. Each row corresponds to an input $x \in \{0,1\}^n$ and the contents correspond to the output $f(x)$.

- Choosing a function f uniformly from $\texttt{Func}_n$ corresponds to choosing each row in the lookup table uniformly and independently of the other rows.

- For a given length-preserving keyed function F_k, choosing k uniformly from $\{0,1\}^n$ induces a distribution over at most 2^n functions with domain and range equal $\{0,1\}^n$.

- The definition of a pseudorandom function will be given with respect to an efficient (polynomial-time) distinguisher D which is given access to an *oracle* $\mathcal{O}$ which is either equal to F_k (for uniform k) or f (for uniform f from $\texttt{Func}_n$). D can query the oracle $\mathcal{O}$ at any point $x \in \{0,1\}^n$ and the oracle returns $\mathcal{O}(x)$. D can adaptively query the oracle but can ask only polynomially many queries.

Definition. *Let F be an efficient, length-preserving, keyed function. F is a* ***pseudorandom function*** *if for all PPT distinguishers D, there is a negligible function* $\texttt{negl}$ *such that:*

$$\left| \Pr\left[D^{F_k(\cdot)}(1^n) = 1\right] - \Pr\left[D^{f(\cdot)}(1^n) = 1\right] \right| \leq \texttt{negl}(n),$$

where the first probability is taken over uniform choice of $k \in \{0,1\}^n$ and the randomness of D, and the second probability is taken over uniform choice of $f \in \texttt{Func}_n$ and the randomness of D.

- D is *not* given access to the key k. If k is known, it is easy to construct a distinguisher which succeeds with non-negligible probability (how?).

- Example of a non-pseudorandom, length-preserving, keyed function: $F(k,x) = k \oplus x$.

4 CPA-Secure Encryption from Pseudorandom Functions

- Let F be a pseudorandom function. Define a private-key encryption scheme for messages of length n as follows:
  - `Gen`: On input 1^n, choose k uniformly from $\{0,1\}^n$.
  - `Enc`: Given $k \in \{0,1\}^n$ and message $m \in \{0,1\}^n$, choose uniform $r \in \{0,1\}^n$ and output the ciphertext
  $$c := \langle r, F_k(r) \oplus m \rangle.$$
  - `Dec`: Given $k \in \{0,1\}^n$ and ciphertext $c = \langle r, s \rangle$, output the plaintext message
  $$m := F_k(r) \oplus s.$$

Theorem. *If F is a pseudorandom function, then the above construction is a CPA-secure private-key encryption scheme for messages of length n.*

Proof. Done in class. □

- What's a drawback of this construction?

5 References and Additional Reading

- Section 3.4, 3.5 from Katz/Lindell